

# Inhomogeneous Fragmentation of the Rolling Tachyon

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## Abstract

Dirac-Born-Infeld type effective actions reproduce many aspects of string theory classical tachyon dynamics of unstable Dp-branes. The inhomogeneous tachyon field rolling from the top of its potential forms topological defects of lower codimensions. In between them, as we show, the tachyon energy density fragments into a p-dimensional web-like high density network evolving with time. We present an analytic asymptotic series solution of the non-linear equations for the inhomogeneous tachyon and its stress energy. The generic solution for a tachyon field with a runaway potential in arbitrary dimensions is described by the free streaming of noninteracting massive particles whose initial velocities are defined by the gradients of the initial tachyon profile. Thus, relativistic particle mechanics is a dual picture of the tachyon field effective action. Implications of this picture for inflationary models with a decaying tachyon field are discussed.

*Introduction.* In this paper we investigate generic inhomogeneous solutions of Dirac-Born-Infeld type theories

$$S = - \int d^{p+1}x V(T) \sqrt{1 + \alpha' \partial_\mu T \partial^\mu T} + \mathcal{O}(\partial_\mu \partial^\mu T) , \quad (1)$$

where  $T(x^\mu)$  is a (dimensionless) scalar field and  $V(T)$  is its runaway potential (no minima). In string theory  $\alpha'$  is a square of the fundamental length scale; we put  $\alpha' = 1$ . The action (1) was proposed in [1] as an effective field theory description of the open string theory tachyon which describes unstable non-BPS D-branes. In application to the string theory tachyon (1) should be understood in the truncated approximation, i.e. valid only in the regime where higher derivatives are not large. The potential is often chosen to be  $V(T) = \tau_p / \cosh T$  for the bosonic case which we consider. At large  $T$  the potential has a runaway character  $V(T) \simeq e^{-T}$  with the ground state at infinity.

There are several motivations for studying the properties of the effective action (1). It is difficult to find the open string tachyon dynamics for generic tachyon inhomogeneities. The action (1), meanwhile, permits us to study complicated tachyon dynamics in terms of classical field theory. The relatively simple formulation of tachyon dynamics in terms of the effective action (1) has therefore triggered significant interest in the investigation of the field theory of the tachyon and the possible role of tachyons in cosmology. Indeed, the end point of string theory brane inflation is annihilation of  $D - \bar{D}$  branes, which leads to the formation and subsequent fragmentation of a tachyon condensate [2]. Thus the potential role of the tachyon in cosmology cannot be understood without first understanding its fragmentation.

Apart from its application to the string theory tachyon, the search for the structure of general solutions of the theory (1) is an interesting mathematical problem in and of itself. The equation of motion arising from the action (1) is an example of a complicated, non-linear, partial differential equation which, as we will show, admits a relatively simple, general, inhomogeneous solution. The evolution of the tachyon field  $T(t, \vec{x})$  can be viewed as a mapping  $T(t_0, \vec{x}_0) \rightarrow T(t, \vec{x})$  that becomes multivalued and generates singularities at caustics [3]. Besides the DBI type theories (1), there are other cosmologically motivated phenomenological models of fields with high derivatives which share the problematics of (1).

If the potential  $V(T)$  is symmetric around  $T = 0$  and the inhomogeneous tachyon field begins rolling from the top of its effective potential, then topological defects (kinks) can form due to symmetry breaking [4]. In this paper we consider what happens to the tachyon

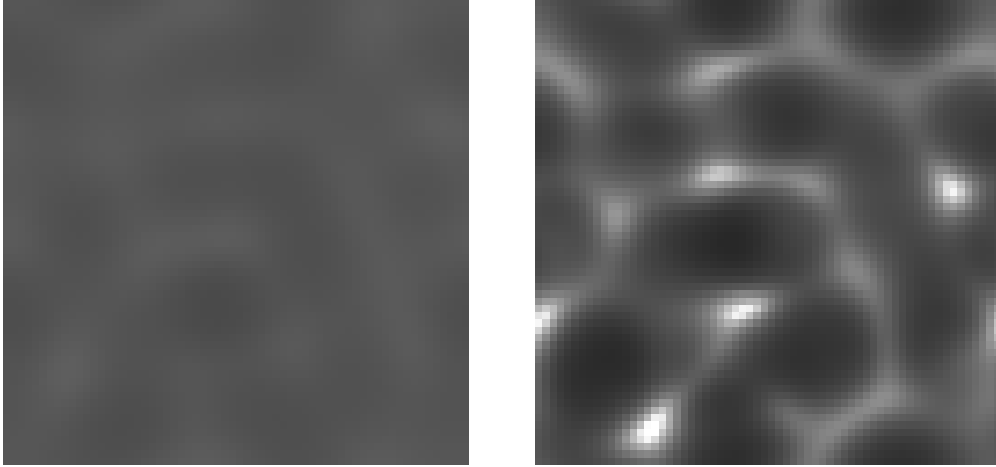


FIG. 1: The focusing of energy density into a web-like structure due to caustic formation. Whiter regions correspond to higher densities. The left panel shows a nearly homogeneous initial Gaussian random field profile and the right panel shows the same field a short time later.

field in the region where it rolls down one side of the potential. We will see the formation of sharp features in the tachyon energy density due to its fragmentation. These features, which are related to the convergence of characteristics of the field  $T$ , have to be distinguished from topological defects. The full picture must incorporate both effects, formation of kinks and tachyon fragmentation in the space between them. Because it is hard to study with CFT tachyon dynamics with a generic, spatially varying profile [5], previous calculations dealt with a plane wave tachyon profile [6]. In this case the tachyon decays into equidistant plane-parallel singular hypersurfaces of co-dimension one, which were interpreted as kinks. The effective action for a plane wave tachyon predicts similar result, as we will see later. However, this inhomogeneous profile is atypical in the sense that fragmentation between kinks does not occur. In the general case we expect both structures, weblike fragmentation and topological defects.

In this paper we concentrate on tachyon fragmentation between kinks. We begin by showing an image that illustrates the fragmentation of the tachyon field as it rolls down one side of its potential. Figure 1 shows the result of a numerical lattice simulation of the energy density of a two dimensional tachyon field rolling down one side of its potential, as described by the equation of motion (2) below. We used the LATTICEASY code [7] adapted for eq. (2). Starting from an initial random Gaussian field  $T$  the energy rapidly became fragmented

into an anisotropic structure of clumps joined by filaments into a web-like network.

The tachyon energy density pattern in Fig. 1 is reminiscent of the illumination pattern at the bottom of the swimming pool or the web-like large scale structure of the universe. The similarity is not coincidental: the underlying mathematics has common features in all three cases. In what follows we present an analytic solution to the tachyon equation of motion that describes in detail the formation of this structure. We begin by describing a good approximation to the dynamics of eq. (2) and go on to show how to extend this approximation into an asymptotic series for the field  $T$ .

*The Free Streaming Approximation.* The equation of motion for the tachyon field follows from the action (1)

$$\partial_\mu \partial^\mu T - \frac{\partial_\mu \partial_\nu T}{1 + \partial_\alpha T \partial^\alpha T} \partial^\mu T \partial^\nu T - \frac{V_{,T}}{V} = 0 . \quad (2)$$

For simplicity we will confine ourselves to a pure exponential potential  $V(T) = e^{-T}$ , however our results are qualitatively valid for any runaway potentials. The energy density of the tachyon field  $\rho = T_{00}$  is

$$\rho = \frac{e^{-T}}{\sqrt{1 + \partial_\mu T \partial^\mu T}} \dot{T}^2 + e^{-T} \sqrt{1 + \partial_\mu T \partial^\mu T} , \quad (3)$$

We have observed solving (2) numerically [3] that if we define an operator  $P(T) = 1 + \partial_\mu T \partial^\mu T$  the field  $T$  rapidly approaches a regime in which  $P(T) \approx 0$ . We write this by saying

$$T(x^\mu) \approx S(x^\mu) , \quad (4)$$

where  $S$  satisfies the equation

$$\dot{S}^2 - \left( \vec{\nabla}_x S \right)^2 = 1. \quad (5)$$

The dot represents a time derivative and the spatial derivatives are with respect to the  $p$  spatial coordinates  $\vec{x}$  on the brane. This equation is the Hamilton-Jacobi equation for the evolution of the wave front function of free streaming massive relativistic particles.

Let us consider this particle description. At some initial time  $t_0$  we can label the position of each particle with a vector  $\vec{q}$ . Equivalently we can say that  $\vec{q}$  parametrizes the different particles. The initial four-velocity of the particle is given by  $\partial^\mu S_0$ . If we further define the proper time  $\tau$  along each particle's trajectory, we can switch from coordinates  $(t, \vec{x})$  to  $(\tau(t, \vec{x}), \vec{q}(t, \vec{x}))$  and obtain an exact parametric solution to (5) [3]

$$\vec{x} = \vec{q} - \vec{\nabla}_{\vec{q}} S_0 \tau \quad (6)$$

$$t = \sqrt{1 + |\nabla_{\vec{q}} S_0|^2} \tau \quad (7)$$

$$S = S_0 + \tau. \quad (8)$$

The interpretation of the solution (6) is very simple and intuitive. It tells us that the field  $S$  propagates along the trajectories of the massive relativistic particles, growing linearly with proper time. The slope of each characteristic depends only on the initial gradients of  $S_0$  on that characteristic.

In geometrical optics photons are massless, but the qualitative picture of their wave front propagation is similar. This explains the similarity between the two dimensional web-like pattern of Fig. 1 and the illumination pattern at the bottom of a swimming pool. The focusing of particle trajectories corresponds to higher density concentrations and further, to the formation of caustics at the loci where trajectories cross.

*The Full Solution.* Let us now consider the energy density. Looking at equation (3) we see that the exponential pieces are growing exponentially small, as are the arguments of the square roots. The second term in (3) will thus rapidly become irrelevant and we need consider only how the exponentially small numerator and denominator of the first term will be related. (The leading term in  $\dot{T}^2$  is one).

To calculate  $\rho$  we will need to go beyond the free streaming approximation (4). In view of the exponential in the numerator of  $\rho$  we conjecture that  $T$  can be expanded as

$$T(x^\mu) \approx S + f_1[S] e^{-2S}, \quad (9)$$

where  $f_1$  is a sub-exponential functional of  $S$ . (We could include a lower order term  $f_0 e^{-S}$ . As we explain below, we can solve exactly for  $f_0$  in that case and we find that its effects can be absorbed into  $S$  and  $f_1$ .) Now we are going to check the validity of this expansion.

Plugging the expansion (9) into equation (2) and keeping only terms proportional to  $e^{-2S}$  gives the following equation for the sub-exponential function  $f_1$

$$(\partial^\mu S \partial^\nu S) \partial_\mu \partial_\nu f_1 + 2(1 - \square S) \partial^\mu S \partial_\mu f_1 - 4 \square S f_1 = 0, \quad (10)$$

where  $\square S = \partial_\nu \partial^\nu S = -\ddot{S} + \nabla_{\vec{x}}^2 S$ .

This equation can be dramatically simplified by changing from  $(t, \vec{x})$  coordinates to  $(\tau, \vec{q})$  coordinates, as defined by the characteristics of  $S$  in (6-7). In these coordinates,  $f_1$  has no spatial derivatives and equation (10) reduces to

$$f_{1,\tau\tau} - 2(1 - \square S) f_{1,\tau} - 4 \square S f_1 = 0, \quad (11)$$

Note that  $\square S$  can be calculated either with respect to  $(t, \vec{x})$  or  $(\tau, \vec{q})$  coordinates. We can further simplify this equation by introducing a new variable  $y$

$$y \equiv 2f_1 - f_{1,\tau} . \quad (12)$$

Equation (11) can be rewritten in terms of  $y$

$$y_{,\tau} + 2\square S y = 0, \quad (13)$$

which can be immediately solved to give  $y(\tau, \vec{q}) = y_0(\vec{q}) \exp(-2 \int^\tau d\tau' \square S)$ . From this and (12)

$$f_1(\tau, \vec{q}) = f_{1i}(\vec{q}) e^{2\tau} \int^\tau d\tau' e^{-2\tau' - 2 \int^{\tau'} d\tau'' \square S} . \quad (14)$$

To proceed further we need to calculate  $\square S$ . In principle it can be done from the solution (6) in parametric form by inverting the  $(t, \vec{x}) \rightarrow (\tau, \vec{q})$  coordinate transformation matrix [12]. Instead we will use the following trick. Plugging the expansion (9) into the energy density (3) we find  $\rho \approx 1/\sqrt{2(2f_1 + \partial_\mu S \partial^\mu f_1)}$ . Observe that the denominator here is *exactly identical* to  $\sqrt{2y}$ , thus

$$\rho \approx \frac{1}{\sqrt{2y}} = \rho_0(\vec{q}) e^{\int^\tau d\tau' \square S} . \quad (15)$$

This is precisely the expression for the energy density of free streaming relativistic particles which obey the relativistic continuity equation  $\partial_\mu S (\rho \partial^\mu S) = 0$ . In fact, (15) is the solution of this continuity equation in the coordinates  $(\tau, \vec{q})$ . However, there is another form of the energy density, which is equivalent to (15)

$$\rho = \frac{\rho_0(\vec{q})}{|\frac{\partial \vec{x}}{\partial \vec{q}}|} , \quad (16)$$

where the denominator is the Jacobian of the  $\vec{x} \rightarrow \vec{q}$  transformation. Indeed, from conservation of the energy density in a differential volume we have  $d^p \vec{q} \rho_0(\vec{q}) = d^p \vec{x} \rho$ , which leads to the formula (16). It is now straightforward to calculate the Jacobian from the formulas (6-7). In  $p$  dimensions it is a polynomial in  $\tau$  of order  $p$ . For example, in the one dimensional case  $|\frac{\partial \vec{x}}{\partial \vec{q}}| = (\tau_c - \tau)$ , where  $\tau_c(\vec{q})$  is a function of the gradients of  $S_0$ . In the two dimensional case  $|\frac{\partial \vec{x}}{\partial \vec{q}}| = (\tau_{c1} - \tau)(\tau_{c2} - \tau)$ , where  $\tau_{cn}$  are function of  $\nabla_{\vec{q}} S_0$ .

Comparing expressions (15) and (16), we can find  $\square S$  and calculate  $f_1$  using (14). We find that a constant originating from the integration in (14) can be absorbed in  $S$  while the rest of  $f_1$  will be a  $2p$  order polynomial in  $\tau$ . This demonstrates the validity of the approximation

(9) by showing that the  $f_1$  term provides only exponentially suppressed corrections to the leading term.

We could go further and include other powers of  $e^{-S}$  in our expansion (9).

$$T(x^\mu) \approx S + \sum_{n=0}^{\infty} f_n[S] e^{-(n+1)S} . \quad (17)$$

We have explicitly checked all such terms up through  $e^{-4S}$ , including a possible term proportional to  $e^{-S}$ , and found that they simply provide corrections to the integration constants of  $S$  and  $f_1$  plus terms that are exponentially suppressed relative to the ones we have discussed. We can further show that all such terms  $f_n$  have the same characteristics as  $S$ , and we therefore conclude that the general inhomogeneous solution  $T$  propagates along the characteristics (6-7). Up to exponentially small corrections (which could in principle be calculated order by order), the complete solution for  $T$  can be represented by the two functions  $S_0(\vec{q})$  and  $f_{1i}(\vec{q})$ .

With these results, we can evaluate the energy density (16) to leading order using only  $f_1$ . For an arbitrary brane dimension  $p$  we have

$$\rho \approx \rho_0 \prod_{n=1}^p (1 - \lambda_n t)^{-1} , \quad (18)$$

where  $\lambda_n(\vec{q})$  are the eigenvalues of  $\partial_\mu (\dot{S}_0^{-1} \partial_\nu S_0)$ . From here we see that the energy density first reaches large values in regions where  $\lambda_n(\vec{q})$  is maximal. For some critical trajectories  $\vec{q}_c$  at a critical time  $t_c$  the energy density becomes singular, which corresponds to caustic formation. This is exactly what one would expect in the picture of free streaming, massive, relativistic particles, where the energy density blows up at the orbit crossings.

*Conclusions.* Our most important conclusion is that the general inhomogeneous solution of the field theory (1) very rapidly approaches the asymptotic form (17), which is equivalent to the relativistic mechanics of freely propagating massive particles with velocities  $\vec{v}_a = -\nabla_{\vec{q}} S_0(\vec{q}_a)$ . In other words, there is a duality between the two Lagrangians

$$V(T) \sqrt{1 + \partial_\mu T \partial^\mu T} \Longleftrightarrow \sum_a \sqrt{1 - \vec{v}_a^2} . \quad (19)$$

The whole process of unstable  $Dp$  brane decay in the the dual picture is described as “crumbling to dust” of massive particles. A complete discussion of the interpretation of massive particles and anisotropic high density structures (clumps, filaments, sheets) which they form

would be beyond the scope of this paper. In the  $1 + 1$  case the high density regions of the orbit crossing were conjectured to be  $D0$  branes [8].

From the velocities of the characteristics (6) we can see that around maxima of the initial field profile the characteristics will tend to diverge and the profile will flatten. In regions around minima, however, characteristics will tend to converge, the profile will become sharper, and after some critical time  $t_c$  the field solution will become multivalued. In the dual picture of relativistic particles this corresponds to caustic formation. At the caustics the energy density blows up. Caustic formation also entails divergences in the second derivatives of  $T$ , which signal the breakdown of the truncated approximation (1). In short the Lagrangians (19) are unable to describe the field  $T$  when its solution becomes multi-valued. In the picture of freely moving massive particles we can include interactions to cause them to stick together as their trajectories intersect with impact parameter  $\sim \sqrt{\alpha'}$ .

Let us make a remark about a one-dimensional plane wave tachyon profile  $T = \cos(x)$  rolling from the top of its (symmetric) effective potential. Since the parts of the field that roll to the right have no minima they do not form caustics, and by symmetry the parts rolling to the left do not either. In this particular case the tachyon fragments into kinks only.

We can also consider a more general profile, however. Tachyonic instability occurs for all inhomogeneous modes  $k$  for which the effective mass  $m^2 = k^2 - 1/\alpha'^2$  is negative. Therefore the generic tachyon initial profile is a superposition of a number of modes, which produces a random Gaussian field  $T_0(\vec{q})$ . These initial conditions typically arise from quantum fluctuations during symmetry breaking (see e.g. [9]). In this case we once again expect the formation of topological defects due to symmetry breaking. Outside these defects, however, the tachyon field will fragment into a web like structure as shown in Fig. 1. If the defects are walls these webs will form within each domain; if they are strings the web of caustics will be mixed in with the strings.

If the spectrum of  $T_0(\vec{q})$  inhomogeneities has scaling properties (as quantum fluctuations do), then the web-like network will evolve in a scaling manner. The smallest scale of the web is defined by the largest tachyonic mode  $k$ . The dual picture of freely moving massive particles which stick together as their orbits intersect gives a simple explanation of such fragmentation.

Finally, we note the relevance of our result for cosmological applications of the tachyon.



In the context of brane inflation, which ends with a pair of  $D3 - \bar{D}3$  branes annihilating, the tachyon is a complex field and strings will be created and the rest of energy is transformed into radiation [10]. If the result of real tachyon field fragmentation which we derive above is extended to the complex tachyon, then annihilation of  $D3 - \bar{D}3$  branes results in the net of strings plus massive particles with the matter dominant equation of state. The absence of radiation domination after brane inflation may pose a problem for the model [11]

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